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Thermal effects in inflation power spectra

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Abstract
If inflation was preceded by a thermal era, then there will be a thermal distribution of gravitons at the time of inflation. Gravitons produced during inflation will be amplified by stimulated emission into the existing thermal gravitons. This will result in an enhancement of the B-mode signal at large angles and which could be observable in WMAP and Planck experiments. Non-observation of this signal will have important implications on standard inflation models and will rule out warm inflation.

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1. Introduction

If inflation [1] was preceded by a radiation era then there would be a thermal background of gravitons at the time of inflation. This thermal distribution of gravitons would have decoupled close to the Planck era. The generation of tensor perturbation during inflation [2] would be by *stimulated emission* [3, 4] into this existing thermal background of gravitational waves. This process changes the power spectrum of tensor modes by an extra temperature-dependent factor $\coth(k/2T)$. At large angular scales ($l \leqslant 30$) the power spectrum $P_T = A_T k^{n_T}$ of gravitational waves generated during inflation would have a spectral index $n_T = -1 - 2\epsilon$, instead of the standard slow roll inflation prediction $n_T = -2\epsilon$ which implies that $C^{\mathrm{BB}}_{l=3} \simeq 10 \times C^{\mathrm{BB}}_{l=30}$. If a thermal enhancement of low l BB modes exists it will be observable with WMAP [5] or in upcoming the Planck [6] experiment.

2. Thermal enhancement of tensor power

The tensor perturbations have two independent degrees of freedom which can be chosen as h^+ and $h^\times$ polarization modes. To compute the spectrum of gravitational waves $h(\mathbf{x}, \tau)$ during inflation, we express $h^{(+)}$ and $h^{(\times)}$ in terms of the creation–annihilation operator

$$h^{(i)}(\mathbf{x}, \tau) = \frac{\sqrt{16\pi}}{a(\tau)M_p} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left[a_{\mathbf{k}} f_k(\tau) + a^{\dagger}_{-\mathbf{k}} f^*_k(\tau)\right] \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$
$$\equiv \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} h_{\mathbf{k}}(\tau)\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{1}$$

where $a(\tau)$ is the scale factor, $\mathbf{k}$ is the co-moving wavenumber, $k = |\mathbf{k}|$, and $M_p = 1.22 \times 10^{19}$ GeV is the Planck mass and $i = +, \times$. The power spectrum of the tensor perturbations is defined as

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle \equiv \frac{2\pi^2}{k^3} P_T \delta^3(\mathbf{k} - \mathbf{k}'). \tag{2}$$

The usual quantization condition between the fields and their canonical momenta yields $\left[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}\right] = \delta^3(\mathbf{k} - \mathbf{k}')$ and the vacuum satisfies $a_{\mathbf{k}}|0\rangle = 0$. If the graviton field had zero occupation prior to inflation then $\left[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}\right] = \delta^3(\mathbf{k} - \mathbf{k}')$ and the vacuum satisfies $a_{\mathbf{k}}|0\rangle = 0$. If the graviton field had zero occupation prior to inflation then $\langle a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} \rangle = 0$ and we would obtain a correlation function $\sim | f_k(\tau)|^2$. However if the graviton field was in thermal equilibrium at some earlier epoch it will retain its thermal distribution even after decoupling from the other radiation fields and its occupation number will be given by

$$\langle a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}'} \rangle = \left(\frac{1}{\mathrm{e}^{k/T} - 1}\right) \delta^3(\mathbf{k} - \mathbf{k}'). \tag{3}$$

Using equations (1) and (3) it can be seen that

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = \frac{16\pi | f_k(\tau)|^2}{a^2(\tau)M_p^2} \left(1 + \frac{2}{\mathrm{e}^{\frac{k}{T}} - 1}\right) \delta^3(\mathbf{k} - \mathbf{k}'),$$
$$= \frac{16\pi | f_k(\tau)|^2}{a^2(\tau)M_p^2} \coth\left[\frac{k}{2T}\right] \delta^3(\mathbf{k} - \mathbf{k}'). \tag{4}$$

From the defining relation, equation (2), for the tensor power spectrum and equation (4) we find that the power spectrum for the thermal inflatons can be expressed in terms of the mode functions $f_k(\tau)$ as

$$P_T(k) = \frac{8k^3}{\pi M_p} \frac{| f_k|^2}{a^2(\tau)} \coth\left[\frac{k}{2T}\right]. \tag{5}$$

The mode functions $f_k(\tau)$ obey the minimally coupled Klein–Gordon equation

$$f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0. \tag{6}$$

In a quasi de Sitter universe during inflation, conformal time τ ($\mathrm{d}\tau \equiv \mathrm{d}t/a$) and the scale factor during inflation $a(\tau)$ are related by $a(\tau) = -1/H\tau(1 - \epsilon)$, where $\epsilon = \frac{M_p^2}{16\pi}\left(\frac{V'}{V}\right)^2$ and V is the potential for the inflaton field.

For constant ϵ the mode functions $f_k(\tau)$ obey the minimally coupled Klein–Gordon equation [7, 8],

$$f_k'' + \left[k^2 - \frac{1}{\tau^2}\left(\nu^2 - \frac{1}{4}\right)\right] f_k = 0, \tag{7}$$

where $k = |\mathbf{k}|$ and, for small ϵ and δ, $\nu = \frac{3}{2} + \epsilon$. Equation (7) has the general solution given by

$$f_k(\tau) = \sqrt{-\tau}\left[c_1(k)H^{(1)}_{\nu}(-k\tau) + c_2(k)H^{(2)}_{\nu}(-k\tau)\right]. \tag{8}$$

When the modes are well within the horizon they can be approximated by flat spacetime solutions $f_k{}^0(\tau) = \frac{1}{\sqrt{2k}}\,\mathrm{e}^{-\mathrm{i}k\tau}$, $(k \gg aH)$. Matching the general solution in equation (8) with the solution in the high frequency (flat spacetime) limit gives the value of the constants of integration $c_1(k) = \frac{\sqrt{\pi}}{2}\,\mathrm{e}^{\mathrm{i}(\nu+\frac{1}{2})\frac{\pi}{2}}$ and $c_2(k) = 0$. Equation (8) then implies that for $-k\tau \gg 1$ or $k \ll aH$,

$$f_k(\tau) = \mathrm{e}^{\mathrm{i}(\nu-\frac{1}{2})\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)}\frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu}. \tag{9}$$

Substituting the solution as given in equation (9) for the super-horizon modes in $(k \ll aH)$ in equation (5) for the tensor power spectrum, we obtain

$$P_T(k) = \frac{16\pi}{M_P^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_T}\coth\left[\frac{k}{2T}\right], \tag{10}$$

with $n_T = 3 - 2\nu = -2\epsilon$. We can now rewrite the power spectrum as

$$P_T(k) = A_T(k_0)\left(\frac{k}{k_0}\right)^{n_T}\coth\left[\frac{k}{2T}\right], \tag{11}$$

where k_0 is referred to as the pivot point and $A(k_0)$ is the normalization constant, $A_T(k_0) = \frac{16\pi}{M_P^2}\left(\frac{H_{k_0}}{2\pi}\right)^2$, where H_{k_0} is the Hubble parameter evaluated when $aH = k_0$ during inflation.

The angular power spectrum of the BB polarization modes generated by the gravitational waves is given by [9],

$$C_l^{\mathrm{BB}} = (4\pi)^2\int \mathrm{d}k\,k^2 P_T(k)\left|\int \mathrm{d}\eta\, g(\eta)h_k(\eta)\left[2j_l''(x) + \frac{4j_l(x)}{x}\right]\right|^2, \tag{12}$$

where $g(\eta) = \dot{\kappa}\,\mathrm{e}^{-\kappa}$ is the visibility function and $\dot{\kappa}$ is the differential optical depth for the Thomson scattering. The EE polarization also gets a contribution from the tensor perturbations but it is dominated by the scalar perturbations. So the best signal for gravitational waves is the BB polarization angular spectrum which is generated by the primordial tensor perturbations only.

The temperature-dependent factor becomes important when the ratio $k/(2T)$ is less than the unity. The co-moving wave-number k and the co-moving temperature T can be related to the physical parameters at the time of inflation as follows. Taking the largest measurable perturbation scale $k_{\mathrm{now}}/a_{\mathrm{now}} \simeq R_h^{-1}$ (where $R_h = 4000$ Mpc is the size of the present horizon) and assuming that perturbations of the present horizon scale were just leaving the inflationary horizon H^{-1} at the beginning of inflation, we see that the temperature at the beginning of inflation T_i/a_i must be

$$\frac{k}{2T} = \frac{Ha_i}{2T_i} < 1, \tag{13}$$

in order to have a significant effect on the tensor power spectrum. $T_i/a_i \sim (30V/g_*\pi^2)^{1/4}$, V being the inflaton potential which is related to the curvature at the time of inflation, $H = (8\pi/3)^{1/2}V^{1/2}/M_P$ and $g_* \sim 100$ is the effective number of relativistic particles. Therefore, inflation is expected to start at a temperature $T_i/a_i = 0.24(HM_P)^{1/2}$. Actually the gravitons which are decoupled will have a temperature slightly below the radiation temperature because of the particles (like the inflaton itself) which have annihilated into radiation prior to inflation. But as the effective number of particles $g_* \sim 100$ is large, this difference of temperature is not significant. So for inflation at the GUT scale, $V^{1/4} \sim 10^{15}$ GeV, we have $H \sim 10^{11}$ GeV and the temperature at the start of inflation $T_i/a_i \sim 10^{14}$–10^{15} GeV. So the enhancement of the graviton power spectrum by the factor $\coth\left(\frac{k}{2T}\right) = \coth\left(\frac{Ha_i}{2T_i}\right)$ could be by as large as a factor of 10^4–10^5 at low k due to thermal gravitons.

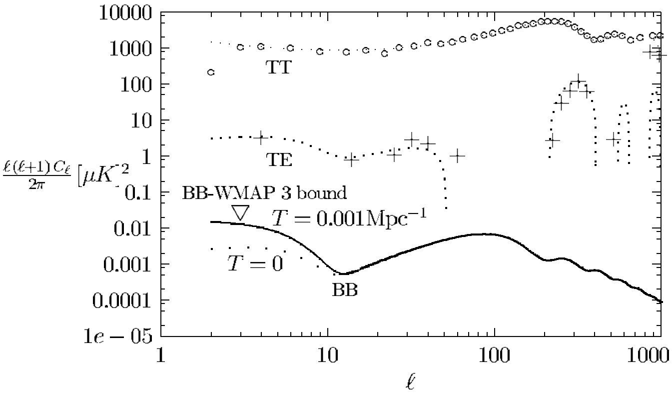

Figure 1. The TT, TE and the BB correlations with thermal graviton spectrum along with the WMAP three years data [5]. The plots for TT, TE and BB correspond to co-moving graviton temperature $T = 0.001$ Mpc^{-1}. For comparison we have plotted the BB angular correlations at $T = 0$. We see that with a graviton temperature $T = 0.001$ Mpc^{-1} the BB correlations are amplified at $l < 30$.

In figure 1, we show the angular correlations of CMBR temperature and polarization assuming a thermal graviton spectrum (along with the WMAP three years data [5]). The plots for TT, TE and BB correspond to co-moving graviton temperature $T = 0.001$ Mpc^{-1}. For comparison we have plotted the BB angular correlations at $T = 0$. We see that with a temperature $T = 0.001$ Mpc^{-1} the BB correlations are amplified at $l < 30$. We see that only the BB correlation is enhanced by the correction to the tensor power spectrum as expected. The contribution of tensors to the TT angular spectrum is comparable at low l to the contribution from the scalars, and there exists the possibility that this large tensor contribution at low l may be detected from the analysis of the TT angular spectrum alone.

We have added the unlensed scalar and tensor contributions to generate the TT, EE, TE and BB correlations. The plots were obtained by running CMBFAST [10], with the following parameters: $\Omega_b = 0.05$, $\Omega_c = 0.25$ and $\Omega_v = 0.70$. The optical depth $\tau = 0.08$ and the Hubble parameter $h = 0.7$. The value of the scalar spectral index $n_s = 0.97$ and the value of the tensor spectral index is taken $n_T = -0.01$. The tensor to scalar ratio is taken to be $r(k_0) = 0.1$ at $k_0 = 0.002$ Mpc^{-1}. The output of the CMBFAST was normalized to the WMAP values at $k = 0.002$ Mpc^{-1} (i.e. $l = 30$). For the curves shown in figure 1 the tensor power spectra is modified due to thermal effects with $\frac{k}{2T} = 500k$. At $k = 0.0002$ Mpc^{-1}, $\coth(500k) = 10$, so there is a large enhancement of the BB polarization at $l = 2 - 6$, while at $k_0 = 0.002$ Mpc^{-1}, $\coth(500k_0) \sim 1.3$, and there is hardly any enhancement of the BB signal (or in the value of $r(k_0)$ in keeping with the observational constraints from WMAP + SDSS [11]). The magnitude of the co-moving graviton temperature needed to produce this effect is $T/a_{\rm now} = 10^{-3}$ Mpc^{-1}. This corresponds to a temperature of $T_i/a_i \simeq 4 \times R_h^{-1} \times a_{\rm now}/a_i = 4H$ (where $R_h \sim 4000$ Mpc is the size of the present horizon). As we have seen inflation can start as soon as the temperature T_i/a_i falls below $V^{1/4} \sim 10^4 H$. Therefore, a temperature larger than $4H$ at the beginning of inflation is not unreasonably high.

2.1. Standard inflation

In standard inflation models the vacuum fluctuations of the inflaton field give the density perturbations. The inflaton is assumed to be decoupled from the radiation at the onset of inflation; however if there was a radiation era prior to inflation then the scalar curvature power spectrum is modified by the same temperature-dependent factor [12] as the tensor power spectrum in (10),

$$P_R(k) = \frac{H^4}{4\pi^2\dot{\phi}^2}\left(\frac{k}{aH}\right)^{n_s-1} \coth\left[\frac{k}{2T}\right]. \tag{14}$$

The extra temperature-dependent term implies that there should be an up-turn of the TT anisotropy spectrum at low l. This expected up-turn in $l(l+1)C_l$ is not seen in the WMAP one-year TT spectrum [12]. This means that there is no significant number density of the background density of inflatons at the time when the modes, which are currently entering our horizon, were exiting the horizon during inflation. This could happen for two reasons. The background density of inflatons may have decayed or annihilated into lighter particles by this time or the inflaton was cooled from the expected temperature of $0.24(HM_P)^{1/2}$ to below H by the time the modes corresponding to our present horizon were leaving the de Sitter horizon. This implies that there were an extra $\Delta N = \ln(0.24(M_P/H)^{1/2})$ e-foldings (which has the value $\Delta N \sim 10$ for GUT scale inflation) than what is needed to solve the horizon problem. In the case of gravitons the first condition does not apply as they decouple at the Planck scale, and if the expected upturn in the BB mode spectrum is not seen that would imply that the duration of inflation was longer than what is needed to solve the horizon problem.

2.2. *Warm inflation*

In warm inflation models [13] where the inflaton is in thermal equilibrium with the radiation bath and the scalar curvature perturbations are generated by thermal fluctuations instead of quantum fluctuations so there is no $\coth(k/2T)$ correction in the inflaton power spectrum due to stimulated emission. However, this correction factor will be present in the graviton spectrum since gravitons are still produced by quantum fluctuations. The scalar curvature perturbation in warm inflation is [14]

$$P_{\mathcal{R}}^{(\mathrm{warm})} = \left(\frac{\pi}{4}\right)^{1/2} \frac{H^{5/2}\Gamma^{1/2}T_r}{\dot{\phi}^2}, \tag{15}$$

where Γ designates the decay width of the inflaton field and T_r is the temperature of the radiation bath.

There are observational constraints on the tensor–scalar ratio

$$r(k_0) = \frac{P_T(k_0)}{P_{\mathcal{R}}(k_0)}. \tag{16}$$

From the combination of WMAP three year data [11] and SDSS large scale structure surveys [15] we have the bound $r(k_0 = 0.002\ \mathrm{Mpc}^{-1}) < 0.28(95\%CL)$, where $k_0 = 0.002\ \mathrm{Mpc}^{-1}$ corresponds to $l = \tau_0 k_0 \simeq 30$ with the distance to the decoupling surface $\tau_0 = 14\,400$ Mpc. SDSS measures galaxy distributions at red shifts $a \sim 0.1$ and probes k in the range $0.016h\ \mathrm{Mpc}^{-1} < k < 0.11h\ \mathrm{Mpc}^{-1}$. From the expressions of $P_{\mathcal{R}}$ in warm inflation, equation (15), and P_T we see that the scalar–tensor ratio in warm inflation models (assuming a nearly scale invariant tensor power spectrum) has a scale dependence at large angles given by

$$r(k) \simeq r(k_0)\frac{\coth\left[\frac{k}{2T}\right]}{\coth\left[\frac{k_0}{2T}\right]} \simeq r(k_0)\left(\frac{k_0}{k}\right). \tag{17}$$

We see that $r(k)$ has a spectral index $n_T \sim -1$ for large scale perturbations. If we consider $k \sim 0.0002\ \mathrm{Mpc}^{-1}$ which corresponds to $l \sim 3$ then the value of $r(k) = 10r(k_0)$. So even with $r(k_0) \sim 0.1$ as constrained by galaxy surveys, we can have $r(k) \simeq 1$ at the quadrupole anisotropy. The B mode polarization at $l = 3$ is enhanced from its value at $l = 30$ by a corresponding factor of 10. This is true as long as the temperature $T_i/a_i \leqslant 10^4 H$ which as we have seen in the earlier discussion is expected if there is a thermal era prior to inflation.

For example taking the inflaton potential to be $V = (1/2)m^2\phi^2$, we have the scalar power

$$P_{\mathcal{R}}^{(\mathrm{warm})}(k_0) = 5.3\frac{\Gamma^{5/2}\phi_0^{1/2}T_r}{M_p^{5/2}m^{3/2}}, \tag{18}$$

and the tensor power

$$P_T(k_0) = \frac{128\pi}{9}\frac{m^2\phi_0^2}{M_P^4}\coth\left[\frac{k_0}{2T}\right], \tag{19}$$

and the scalar–tensor ratio

$$r(k_0) = 8.413\left(\frac{m^{7/2}\phi_0^{3/2}}{M_P^{3/2}}\right)\frac{1}{\Gamma^{5/2}T_r}\coth\left[\frac{k_0}{2T}\right], \tag{20}$$

where ϕ_0 is the value of the inflaton field when the scale $k_0 = 0.002$ Mpc^{-1} was leaving the inflaton horizon. By choosing the parameters $m = 1.4\times10^{12}$ GeV, $\Gamma = 0.5\times10^{13}$ GeV, $T \simeq T_r = 0.24\times10^{16}$ GeV, $\phi_0 \simeq 0.8\times10^{19}$ GeV we have $P_R \simeq 2.3\times10^{-9}$ as required by the WMAP three year data and $r(k_0) = 0.095$. The value of r is larger at $k = 0.0002$ Mpc^{-1} by a factor of $\sim$10 and the B modes are magnified at $l = 3$ compared to their value at $l = 30$ by a factor of 10, also in warm inflation scenarios.

3. Discussions and conclusions

Direct observation of gravitational waves would nail the last still unconfirmed prediction of inflation. The amplitude of gravitational waves gives the Hubble curvature during inflation and would tell us the value of the inflation potential [16, 17]. In addition, gravitational waves produced during inflation can have several applications like leptogenesis by the gravitational spin-coupling to neutrinos [18] or by a gravitational Chern–Simon coupling of the lepton number current [19]. Observation of the B mode polarization in the CMB would confirm the existence of primordial super-horizon gravitational waves. Observationally, the three year WMAP data only give an upper bound on C_l^{BB} with $l = (2-6)$ [5]. The error bars on the C_l^{BB} are presently a factor of 5 larger than the predictions from standard inflation theory with scalar–tensor ratio as large as 0.3, which is close to the observational upper bound $r_{0.002} < 0.28(95\%CL)$. In this paper we show that due to thermal gravitons, the C_l^{BB} at low $l \simeq (2-6)$ could be larger by a factor of 10 compared to what would be expected from the observational constraint on r and could be within the range of observability of WMAP. The upcoming Planck experiment [6] will measure $C_{l=(1-10)}^{\mathrm{BB}}$ at the level of $10^{-4}(\mu K)^2$. Ground-based polarization experiments [20] such as QUaD, QUIET, Clover and PolarBear measure anisotropies at small angular scales only (at $l > 100$, where thermal effects discussed in this paper are negligible) and can observe C_l^{BB} at the level $10^{-2}(\mu K)^2$. These experiments can probe r in the range 0.05–0.1 independent of thermal effects. A combination of data from WMAP/Planck at large angles and ground-based polarization experiments at small angles will therefore either observe or definitely rule out the thermal enhancement effect.

If WMAP or Planck rule out a spectral index of $n_T \sim -1$ at low l, which is the prediction from thermal gravitons, then for the standard inflationary models it would mean that the duration of inflation has to be longer by $\Delta N = \ln(0.24(M_P/H)^{1/2})$ e-foldings than what is needed to solve the horizon problem. Warm inflation models [13, 14] cannot evade this constraints by supercooling during inflation. If B modes are observed and the tensor spectral index at low l is not close to -1, then warm inflation models can be ruled out.

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